Riemann Surfaces
Example Sheet 2

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Please send any comments or corrections to me at `hjrw2@cam.ac.uk`.

1. Recall that an open set $U \subseteq \mathbb{C}$ is a *star domain* if there is $z_0 \in U$ such that, for every $z \in U$, the straight-line segment from z_0 to z is contained in U. Prove that every star domain is simply connected.

2. Show that a regular covering map $\pi : \widetilde{X} \to X$ of non-empty, path-connected topological spaces is surjective. Use the monodromy theorem to show that if X is simply connected then π is a homeomorphism.

3. Suppose that $f : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is an analytic map of complex tori and π_j denotes the projection map $\mathbb{C} \to \mathbb{C}/\Lambda_j$ for $j = 1, 2$. Show that there is an analytic map $F : \mathbb{C} \to \mathbb{C}$ such that $\pi_2 \circ F = f \circ \pi_1$.

   [Hint: Define F as follows. Choose a point μ in $\mathbb{C}$ such that $\pi_2(\mu) = f\pi_1(0)$. For $z \in \mathbb{C}$, join 0 to z by a path $\gamma : [0,1] \to \mathbb{C}$, and observe that the path $f \circ \pi_1 \circ \gamma$ in $\mathbb{C}/\Lambda_2$ has a unique lift to a path $\tilde{\gamma}$ in $\mathbb{C}$ with $\tilde{\gamma}(0) = \mu$. If we define $F(z) = \tilde{\gamma}(1)$, show that $F(z)$ does not depend on the path γ chosen and that F has the required properties.]

4. Let f and F be as in Question 3, and suppose that f is a conformal equivalence. Show that $F(z) = \lambda z + \mu$, for some $\lambda \in \mathbb{C}_*$. Hence deduce that two analytic tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are conformally equivalent if and only if the lattices are related by $\Lambda_2 = \lambda\Lambda_1$ for some $\lambda \in \mathbb{C}_*$.

5. Show that two complex tori, $\mathbb{C}/\langle 1, \tau_1\rangle$ and $\mathbb{C}/\langle 1, \tau_2\rangle$, are conformally equivalent if and only if
$$\tau_2 = \pm\frac{a\tau_1 + b}{c\tau_1 + d}$$
for some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

6. Show that the component of the space of germs over $\mathbb{C}_*$ corresponding to the complex logarithm is analytically isomorphic to the Riemann surface constructed by gluing, and hence also analytically isomorphic to $\mathbb{C}$. Show that the component of the space of germs over $\mathbb{C} \setminus \{-1, 0, 1\}$ corresponding to the complete analytic function $(z^3 - z)^{1/2}$ is analytically isomorphic to the Riemann surface we constructed by gluing (see Example 6.3 in the printed notes).

7. Let R denote the Riemann surface associated with the complete analytic function $\sqrt{1-\sqrt{z}}$ over $\mathbb{C}_*$. Show that the projection covering map to $\mathbb{C}_*$ is surjective. Find analytic continuations along homotopic curves in $\mathbb{C}_*$, say from $1/2$ to $3/2$, which have the same initial germ at $1/2$ but different final germs at $3/2$. Why is this consistent with the classical monodromy theorem?

8. Consider the analytic map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by the polynomial $z^3 - 3z + 1$; find the ramification points of f and the corresponding ramification indices. What are the branch points?

9. Suppose that $f : R \to S$ is an analytic map of compact Riemann surfaces, and let $B \subset S$ denote the set of branch points. Show that the map $f : R \setminus f^{-1}(B) \to S \setminus B$ is a regular covering map. [Hint: Mimic the proof of the valency theorem.] Given a point $P \in S \setminus B$, explain how a closed curve γ in $S \setminus B$ starting and ending at P defines a permutation of the (finite) set $f^{-1}(P)$. Show that the group obtained from all such closed curves is a transitive subgroup of the full symmetric group of the fibre $f^{-1}(P)$. What group is obtained in Question 8?

10. Let $f(z) = p(z)/q(z)$ be a rational function on $\mathbb{C}$, where p, q are coprime polynomials. Show that f defines an analytic map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, whose degree d is the maximum of the degrees of p and q. If f' denotes the derivative of the function f, show that it defines an analytic map $f' : \mathbb{C}_\infty \to \mathbb{C}_\infty$, whose degree satisfies $d-1 \le \deg f' \le 2d$. [Hint: Consider the principal parts of f at its poles.] Give examples to demonstrate that the bounds can be achieved.

11. If $f : R \to S$ is a non-constant analytic map of compact Riemann surfaces, show that their genera satisfy $g(R) \ge g(S)$. Show that any non-constant analytic map between compact Riemann surfaces of the same genus $g > 1$ must be an analytic isomorphism. Does this last statement hold when $g = 0$ or 1?

12. Let $\pi : R \to \mathbb{C} \setminus \{1, i, -1, -i\}$ be the Riemann surface associated to the complete analytic function $(z^4 - 1)^{1/4}$. Describe R explicitly by a gluing construction. Assuming the fact that R may be compactified to a compact Riemann surface $\overline{R}$ and π extended to an analytic map $\bar{\pi} : \overline{R} \to \mathbb{C}_\infty$, find the genus of $\bar{R}$.